Math 53: Multivariable Calculus **Sections 201, 204**

Worksheet for 2021-09-20

Conceptual questions

Question 1. Suppose $f(x,y)$ is a function and $P \in \mathbb{R}^2$ is a point such that $\nabla f(P) = \langle 5, 12 \rangle$. What number should be put in the blank below

$$\langle 5, 12, \_\_\_ \rangle$$

to get a normal vector for the tangent plane to $z = f(x,y)$ at the point P? How about a vector parallel to that plane?

Question 2. Suppose $g(x,y)$ is a function, $\mathbf{u} = \langle 3/5, 4/5 \rangle$ and $Q \in \mathbb{R}^2$ is a point such that $D_{\mathbf{u}} g(Q) = -3$. What number should be put in the blank below

$$\left\langle \frac{3}{5}, \frac{4}{5}, \_\_\_ \right\rangle$$

to get a vector parallel to the tangent plane to $z = g(x,y)$ at the point Q? Is there enough information to obtain a normal vector? What if you were also told that $D_{\mathbf{v}} g(Q) = 4$, where $\mathbf{v} = \langle 4/5, 3/5 \rangle$?

Computations

Problem 1. This problem is recycled from the last worksheet (we hadn't covered the chain rule at that time).

(a) Suppose that $\mathbf{r}(t) = (x(t), y(t))$ is parametrized by arclength (recall that this means $|\mathbf{r}'(t)| = 1$; the particle is "moving at speed 1"). Show that the directional derivative of f in the direction of $\mathbf{r}'(t)$ is equal to $\frac{\mathrm{d}}{\mathrm{d}t}(f(\mathbf{r}(t)))$. Hint: Use the chain rule.

(b) Consider the function

$$f(x,y) = \cos^{-1}\left(\frac{x^2 - y^2}{x^2 + y^2}\right).$$

Using the preceding part, compute $f_y(1,0)$. Hint: Use the unit circle. You should not need the derivative of arccosine.

Below are brief answers to the worksheet exercises. If you would like a more detailed solution, feel free to ask me in person. (Do let me know if you catch any mistakes!)

Answers to conceptual questions

Question 1. $\langle 5, 12, -1\rangle$ is a normal vector. $\langle 5, 12, 169\rangle$ is a tangent vector.

Question 2. $\langle 3/5, 4/5, -3\rangle$ is a tangent vector. This is not enough information to obtain a normal vector. However, if we also knew $\langle 4/5, 3/5, 4\rangle$ as a tangent vector, then we could compute their cross product, equal to $\langle 5, -24/5, -7/25\rangle$, which is a normal vector. Of course, any nonzero multiple of this would suffice too.

Answers to computations

Problem 1.

(a) The directional derivative in the direction of $\mathbf{r}'(t)$ is

$$\begin{aligned}\nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t) &= \frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} \\ &= \frac{\mathrm{d}}{\mathrm{d}t} f(\mathbf{r}(t))\end{aligned}$$

where in the last step the chain rule was used.

(b) Use $\mathbf{r}(t) = \langle \cos t, \sin t\rangle$, which is parametrized by arclength and has $\mathbf{r}'(0) = \langle 0, 1\rangle$. So by the preceding part,

$$\begin{aligned} f_y(1,0) &= D_{\langle 0,1\rangle} f(x,y) \\ &= \left(\frac{\mathrm{d}}{\mathrm{d}t} f(\mathbf{r}(t))\right)\Big|_{t=0} \\ &= \left(\frac{\mathrm{d}}{\mathrm{d}t}(2t)\right)\Big|_{t=0} \\ &= 2.\end{aligned}$$